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► To cite this version:

Laurent Boudin, Julien Mathiaud. A numerical scheme for the one-dimensional pressureless gases system. Numerical Methods for Partial Differential Equations, 2012, 28 (6), pp.1729-1746. 10.1002/num.20700 . hal-00537145v2

HAL Id: hal-00537145

<https://hal.science/hal-00537145v2>

Submitted on 28 May 2011

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A NUMERICAL SCHEME FOR THE ONE-DIMENSIONAL PRESSURELESS GASES SYSTEM

LAURENT BOUDIN AND JULIEN MATHIAUD

ABSTRACT. In this work, we investigate the numerical approximation of the one-dimensional pressureless gases system. After briefly recalling the mathematical framework of the duality solutions introduced by Bouchut and James [6], we point out that the upwind scheme for density and momentum does not satisfy the one-sided Lipschitz (OSL) condition on the expansion rate required for the duality solutions. Then we build a diffusive scheme which allows the OSL condition to be recovered by following the strategy described in [9] for the continuous model.

1. INTRODUCTION

During the last two decades, there have been many contributions on the pressureless gases system, and it seems natural to tackle the question of its discretization. The pressureless gases system appears as a system of conservation laws on the mass and momentum. Hence, it is relevant to wonder if standard numerical schemes for conservation laws, like the upwind scheme, for instance, are fitted to this particular system. However, we emphasize that it is a degenerate hyperbolic system (the Jacobian is not diagonalizable).

Let us now recall the one-dimensional system describing a pressureless gas. Let $T > 0$. The gas density $\varrho(t, x) \geq 0$ and the momentum $q(t, x) \in \mathbb{R}$ satisfy the following equations in $(0, T) \times \mathbb{R}$

$$(1) \quad \partial_t \varrho + \partial_x(\varrho u) = 0,$$

$$(2) \quad \partial_t q + \partial_x(qu) = 0.$$

One must define the velocity $u(t, x) \in \mathbb{R}$ as a quotient of q by ϱ , but this may not be possible, since ϱ can be zero. We discuss this issue below, by recalling the notion of duality solutions [5]. As already stated, each equation consists of a conservation law, (1) for mass and (2) for momentum. We obviously need initial conditions

$$(3) \quad \varrho(0, \cdot) = \varrho^{\text{in}}, \quad q(0, \cdot) = q^{\text{in}},$$

in which the condition on the momentum can be replaced by an initial condition on the velocity $u(0, \cdot) = u^{\text{in}}$, and then written again as $q(0, \cdot) = \varrho^{\text{in}} u^{\text{in}}$.

The previous system can be seen as a simplified model of the Euler equations, where the pressure has been set to zero. It can describe either cold plasmas or galaxies' dynamics [27]. This system (1)–(2) and related problems (traffic models, magnetohydrodynamics, astrophysics, pressureless fluid equations...) have been widely studied, see, for instance, [4, 19, 16, 12, 25, 6, 9, 26, 1, 24, 22, 7, 17, 15, 2, 23, 11]. Those references use the same fluid point of view we choose here, or the kinetic one, involving the adhesion dynamics of the so-called sticky particles.

When one studies smooth solutions of the pressureless gases system, (2) can be replaced by the standard Burgers equation:

$$(4) \quad \partial_t u + \partial_x \left(\frac{u^2}{2} \right) = \partial_t u + u \partial_x u = 0.$$

System (1)–(2) is then uncoupled, since we obtain u from (4), and then ϱ from (1). On the other hand, it is well-known that smooth initial data can result in mass concentration, for example, when the velocity does not increase. In that case, the velocity cannot satisfy (4) anymore.

In [5], Bouchut and James introduced the notion of duality solution for one-dimensional transport equations and conservation laws. In [6], they prove that this framework is fitted to the pressureless gases system. Let us briefly recall the results they obtain.

Definition 1. A couple $(\varrho, q) \in C(\mathbb{R}_+; \mathcal{W}^*(\mathcal{M}_{\text{loc}}(\mathbb{R})))^2$, with $\varrho \geq 0$, is a duality solution to (1)–(2), if there exists a bounded Borel function a and $\alpha \in L^1_{\text{loc}}(\mathbb{R}_+^*)$ such that

$$\partial_x a \leq \alpha, \quad q = a\varrho, \quad \text{in } \mathbb{R}_+^* \times \mathbb{R},$$

and, in the duality sense on $(t_1, t_2) \times \mathbb{R}$, for any $0 < t_1 < t_2$,

$$\partial_t \varrho + \partial_x (a\varrho) = 0, \quad \partial_t q + \partial_x (qa) = 0.$$

In that setting, u is defined ϱ -almost everywhere, and we have $u = a$ ϱ -a.e. Bouchut and James prove that duality solutions are stable, and also entropic, i.e. the following inequality holds, in the distributional sense,

$$(5) \quad \partial_t (\varrho S(u)) + \partial_x (\varrho u S(u)) \leq 0,$$

for any convex function S . Using those properties and the sticky particles dynamics, they obtain the following existence result.

Theorem 1. Let $\varrho^{\text{in}}, q^{\text{in}} \in \mathcal{M}_{\text{loc}}(\mathbb{R})$, with $\varrho^{\text{in}} \geq 0$ and $|q^{\text{in}}| \leq U \varrho^{\text{in}}$, $U \geq 0$. Then there exists a duality solution to (1)–(3), and we have $\|a\|_\infty \leq U$ and $\alpha(t) = 1/t$.

As proven in [20], the one-sided Lipschitz (OSL) condition on the expansion rate $\partial_x a \leq 1/t$, also known as the Oleinik entropy condition, is optimal

for a convex scalar conservation law. In the proof of Theorem 1, it is clear that the standard convex entropy condition (5) is not enough, and the OSL condition is really required. Note that, when the solutions are smooth, this estimate can easily be proven, since the Burgers equation (4) lies in the class of convex scalar conservation laws [13, 21].

Eventually, Bouchut and James also obtain uniqueness when ϱ^{in} is nonatomic (essentially meaning that ϱ^{in} is smooth).

In this work, we also consider the viscous pressureless gases system. In this system, as explained in [9], (2) is replaced by an equation on the velocity itself. Let us choose $\varepsilon > 0$. The gas density $\varrho(t, x) > 0$ and the velocity $u(t, x) \in \mathbb{R}$ satisfy, in $(0, T) \times \mathbb{R}$, Equation (1) and

$$(6) \quad \partial_t u + u \partial_x u = \frac{\varepsilon}{\varrho} \partial_{xx}^2 u,$$

with the same set of initial conditions (3). That writing imposes that ϱ remains nonnegative, which is true if ϱ^{in} is also nonnegative, see [9]. Note that (6) is equivalent, when ε is fixed, to

$$(7) \quad \partial_t u + \partial_x \left(\frac{u^2}{2} \right) = \frac{\varepsilon}{\varrho} \partial_{xx}^2 u,$$

if we take into account the smoothness of the viscous velocity given in [9].

In fact, (6) or (7) can also be rewritten as an equation on the momentum, with a viscosity term $\varepsilon \partial_{xx}^2 u$ on the right-hand side,

$$\partial_t(\varrho u) + \partial_x(\varrho u^2) = \varepsilon \partial_{xx}^2 u,$$

which yields (2) when ε goes to 0. In [9], the author proved the existence, in the sense of distributions, of solutions to the viscous system (1), (3) and (7), and that the expansion rate is upper-bounded: $\partial_x u \leq A/(At + 1)$, when $A = \max(\text{ess sup } \partial_x u^{\text{in}}, 0)$ is finite. He also obtains the convergence of the viscous solutions towards the duality solutions to the pressureless gases system when ε vanishes. More precisely, the following convergence result holds.

Theorem 2. *Let $(\varrho_\varepsilon^{\text{in}})$, $(u_\varepsilon^{\text{in}})$ such that, for any $\varepsilon > 0$,*

$$\varrho_\varepsilon^{\text{in}} > 0, \quad \varrho_\varepsilon^{\text{in}} \in L^\infty(\mathbb{R}), \quad \|1/\varrho_\varepsilon^{\text{in}}\|_{L^\infty(\mathbb{R})} \leq C\varepsilon^{-1/4},$$

$$u_\varepsilon^{\text{in}} \in L^1 \cap L^\infty(\mathbb{R}), \quad \|u_\varepsilon^{\text{in}}\|_{L^\infty(\mathbb{R})} \leq C,$$

$$\partial_x u_\varepsilon^{\text{in}} \in L^1 \cap L^2(\mathbb{R}), \quad \text{ess sup } \partial_x u_\varepsilon^{\text{in}} \leq C\varepsilon^{-1/2}.$$

We assume that $(\varrho_\varepsilon^{\text{in}}) \rightharpoonup \varrho^{\text{in}}$ and $(\varrho_\varepsilon^{\text{in}} u_\varepsilon^{\text{in}}) \rightharpoonup q^{\text{in}}$ in $\mathbf{w}^\text{-}\mathcal{M}_{\text{loc}}(\mathbb{R})$. Then, up to a subsequence, $(\varrho_\varepsilon, \varrho_\varepsilon u_\varepsilon)$, given by the solutions to (1) and (7), with initial datum $(\varrho_\varepsilon^{\text{in}}, \varrho_\varepsilon^{\text{in}} u_\varepsilon^{\text{in}})$, converges in $C_t(\mathbf{w}^*\text{-}\mathcal{M}_{\text{loc}}(\mathbb{R}))$ towards the duality solution (ϱ, q) of (1)–(3).*

Both viscous and inviscid systems can also be studied in a periodic framework, *i.e.* we focus on the closed interval $[0, 1]$ and impose that all the physical quantities have the same values at both $x = 0$ and $x = 1$, so that the solutions are 1-periodic.

This work is dedicated to the numerical approximation of the pressureless gases system (1)–(2), where the latter may be replaced by (7). For readability reasons, we choose the periodic framework.

There are two methods to get *a priori* relevant schemes. The first one is to use the natural kinetic framework which underlies the pressureless gas dynamics, with kinetic schemes, as in [4, 8], or with particle methods [14]. The second one is related to the discretization of hyperbolic conservation laws. Gosse and James [18] point out the relevance of two families of numerical schemes: the upwind schemes and the Lax-Friedrichs schemes. In [3], Berthon *et al.* investigate a relaxation scheme for the pressureless gases system in one and two-dimensional settings.

As we already pointed out, the key condition to obtain the duality solution is that the velocity expansion rate must be upper-bounded by $1/t$. Brenier and Osher [13] obtained the relevance of the OSL condition in a discrete framework for the convex scalar conservation laws. In this work, we first investigate the upwind scheme associated to (1)–(2), and prove that it fails to ensure the OSL condition. Subsequently, we try the upwind diffusive scheme associated to (1) and (7), and explain how we can obtain a good numerical approximation of the duality solution to the inviscid pressureless gases system using this scheme. We do not study the Lax-Friedrichs schemes described in [18]. Indeed, the numerical dissipation induced by those first order schemes is too significant. Since it is then natural to use higher order schemes, we recover the same kind of problems as in the diffusive upwind scheme we here propose, involving second order terms.

In the remainder, let $\Delta t, \Delta x > 0$ such that $N = T/\Delta t \in \mathbb{N}^*$ and $I = 1/\Delta x \in \mathbb{N}^*$, and set $\lambda = \Delta t/\Delta x$. We respectively denote ϱ_i^n , q_i^n and u_i^n the approximate values of ϱ , q and u at time $n\Delta t \in [0, T]$ and coordinate $(i + 1/2)\Delta x \in [0, 1)$, for $0 \leq n \leq N$ and $0 \leq i < I$. Since we use a periodic framework, we define ϱ_i^n , q_i^n and u_i^n for any $i \in \mathbb{Z}$, by

$$\varrho_{i+pI}^n = \varrho_i^n, \quad q_{i+pI}^n = q_i^n, \quad u_{i+pI}^n = u_i^n, \quad 0 \leq i < I, \quad p \in \mathbb{Z}^*.$$

For the sake of readability, in the previous notations, we may drop the time iteration index n and replace $n + 1$ by a prime symbol “ $'$ ”. For instance, the velocity at time $(n + 1)\Delta t$ and coordinate $(i + 1/2)\Delta x$ can be written as u'_i or u_i^{n+1} .

Apart from the density, momentum and velocity, the quantity of interest, which we name the numerical expansion rate, is, for each time and space indices n and i ,

$$w_i^n := n\lambda(u_{i+1}^n - u_i^n).$$

Indeed, the OSL condition at time $n\Delta t$ then reads $\max_i w_i^n \leq 1$.

2. UPWIND SCHEME

Let us first denote the positive and negative parts of $a \in \mathbb{R}$

$$a^+ = \max(0, a), \quad a^- = \min(0, a).$$

The upwind scheme writes, for any $0 \leq i < I$,

$$\begin{aligned} \varrho'_i &= \varrho_i - \lambda [\varrho_i(u_i)^+ - \varrho_{i-1}(u_{i-1})^+] - \lambda [\varrho_{i+1}(u_{i+1})^- - \varrho_i(u_i)^-], \\ q'_i &= q_i - \lambda [q_i(u_i)^+ - q_{i-1}(u_{i-1})^+] - \lambda [q_{i+1}(u_{i+1})^- - q_i(u_i)^-], \\ u'_i &= \frac{q'_i}{\varrho'_i}, \quad \text{if } \varrho'_i > 0, \end{aligned}$$

and u'_i is not defined if $\varrho'_i = 0$. It is quite clear that the previous schemes on both ϱ and q are monotonic, if the standard Courant-Friedrichs-Lewy (CFL) condition $\lambda \max |u| \leq 1$ is satisfied. Hence, we only choose positive initial data to study positive velocities. The scheme then becomes, for any $0 \leq i < I$,

$$(8) \quad \varrho'_i = (1 - \lambda u_i) \varrho_i + \lambda u_{i-1} \varrho_{i-1},$$

$$(9) \quad q'_i = (1 - \lambda u_i) q_i + \lambda u_{i-1} q_{i-1},$$

$$(10) \quad u'_i = \frac{q'_i}{\varrho'_i}.$$

As we already stated, this last equality allows to define u'_i only when $\varrho'_i > 0$. This fits the mathematical setting of the pressureless gases system, since u can only be defined ϱ -almost everywhere. Nevertheless, it is not satisfying from a numerical viewpoint, since the computations stop whenever the density becomes equal to 0. We can impose whichever value we want, for instance, $u'_i = 0$, when $\varrho'_i = 0$. Indeed, we do not care about the value of the velocity at a point where there is no matter. But we must keep in mind not to use those artificial nil values of u'_i to study the numerical expansion rate.

Thanks to (10), we immediately have

$$\varrho'_i \varrho'_{i+1} (u'_{i+1} - u'_i) = \varrho'_i q'_{i+1} - \varrho'_{i+1} q'_i,$$

which implies

$$\varrho'_i \varrho'_{i+1} \frac{w'_i}{(n+1)\lambda} = (1 - \lambda u_{i+1}) \varrho_{i+1} \varrho'_i \frac{w_i}{n\lambda} + \lambda u_{i-1} \varrho_{i-1} \varrho'_{i+1} \frac{w_{i-1}}{n\lambda}.$$

Under the CFL condition $\lambda \max |u| \leq 1$, if $(w_i)_{0 \leq i < I}$ are negative, and if $(\varrho'_i)_{0 \leq i < I}$ are nonnegative, it is clear that the quantities (w'_i) also remain negative. Unfortunately, if w_j is nonnegative for a given j , the OSL condition $w'_i \leq 1$ for all i may not be satisfied.

Proposition 3. *Let $0 < \lambda < 1$ and U such that $0 < \lambda U < 1$, and choose an integer $I > 2 + 1/\lambda$. We consider the following initial data*

$$(11) \quad \varrho_i^0 = 1, \quad 0 \leq i \leq I-1, \quad u_0^0 = U, \quad u_i^0 = 0, \quad 1 \leq i \leq I-1.$$

Then the upwind scheme (8)–(10) does not satisfy the OSL condition. More precisely, we have

$$(12) \quad \max_i w_i^{I-2} > U.$$

The assumption on the Courant number $0 < \lambda < 1$ is standard and is a natural consequence of the CFL condition $\lambda U \leq 1$ when U is large. The initial data can of course be defined without the discretization grid: we have $\varrho^{\text{in}} \equiv 1$ and $u^{\text{in}} \equiv 0$ except in 0 where $u^{\text{in}}(0) = U$.

Proof. It is easy to simultaneously prove, by induction on the time index $0 \leq j < I-1$, that

$$\varrho_i^j > 0, \quad 0 \leq i \leq I-1,$$

and

$$u_0^j = U, \quad u_{j+1}^j > 0, \quad u_i^j = 0, \quad j+2 \leq i \leq I-1.$$

We must emphasize that the nil values of u_i^j are obtained because $q_i^j = 0$ and $\varrho_i^j > 0$, and not because of the choice of nil velocity when the density equals 0.

Then we can write that

$$w_{I-1}^{I-2} = (I-2)\lambda(u_0^{I-2} - u_{I-1}^{I-2}) = (I-2)\lambda U,$$

which, together with the choice of I , implies

$$\max_i w_i^{I-2} \geq (I-2)\lambda U > U.$$

Note that the first inequality is in fact an equality, but we do not need to prove it here. \square

Remark 1. As we already pointed out, the standard numerical version of the OSL condition reads $\max_i w_i^n \leq 1$. It may have been relaxed into $\max_i w_i^n \leq K$, where K is a nonnegative constant, which does not depend on the initial data. But (12) implies that the quantity $\max_i w_i^{I-2}$ can be as large as we want, depending on the value of U .

Proposition 3 means in particular that, if the space step Δx is refined enough, the numerical OSL condition cannot be satisfied anymore, with initial data given by (11). Moreover, we must point out that, whatever the final time is, one can find a discretization for which the upwind scheme cannot satisfy the OSL condition, because I does not depend on T .

The initial datum u^{in} in the previous proposition is not smooth. Nevertheless, even with smooth (and periodic) initial data, the upwind scheme does not necessarily provide a solution satisfying the OSL condition, see 4.1.

3. ADDING AN ARTIFICIAL VISCOSITY

As it was done in [9], we now add a small viscosity term in (2) to obtain (7), and we study the numerical approximation of (1) and (7). We still deal with arbitrary 1-periodic initial data $u^{\text{in}} \geq 0$, $\varrho^{\text{in}} \geq 0$.

In what follows, we consider a fixed $\varepsilon > 0$, small enough. If necessary, we regularize both u^{in} and ϱ^{in} so that (keeping the same notations for both, even if they depend on ε) $u^{\text{in}}, \varrho^{\text{in}} \in C^1(\mathbb{R}; \mathbb{R}_+^*)$, remain periodic and satisfy the assumptions of Theorem 2, which can be written as

$$(13) \quad \varrho^{\text{in}}(x) \geq C\varepsilon^{1/4}, \quad u^{\text{in}}(x) \leq C, \quad (u^{\text{in}})'(x) \leq \frac{C}{\sqrt{\varepsilon}}, \quad \forall x \in [0, 1],$$

where C is a constant which does not depend on ε . The regularized ϱ^{in} must lie in \mathbb{R}_+^* , since the continuous diffusive model involves a division by ϱ .

In the following, we set

$$\begin{aligned} U &= \max_{[0,1]} u^{\text{in}} > 0, & V &= \min_{[0,1]} u^{\text{in}} > 0, \\ A &= \max(0, \max_{[0,1]} (u^{\text{in}})') \geq 0, & R &= \min_{[0,1]} \varrho^{\text{in}} > 0. \end{aligned}$$

The previous quantities can depend on ε , and must satisfy properties which come from (13), *i.e.*

$$(14) \quad R \geq C\varepsilon^{1/4}, \quad V \leq U \leq C, \quad A \leq \frac{C}{\sqrt{\varepsilon}},$$

where C does not depend on ε .

Then we consider $\Delta t, \Delta x > 0$, and set

$$\lambda = \frac{\Delta t}{\Delta x}, \quad \sigma = \frac{\Delta t}{\Delta x^2}.$$

In the remainder, we make the following assumptions on the time and space steps:

$$(15) \quad 0 < \Delta x \leq \frac{2V}{1+A},$$

$$(16) \quad 0 < \Delta t \leq \min \left(\frac{1}{4A+1}, \frac{1}{4U} \Delta x, \frac{R}{4\varepsilon(1+AT)} \Delta x^2 \right).$$

In fact, (15) and (16) are not so restrictive, since, eventually, Δx and Δt will go to 0, ε being fixed. From now on, even if we do not write the dependence on ε , we must keep in mind, in the numerical examples, that U , V , A and R can depend on ε and must satisfy (14), at least for ε small enough. That dependence implies that, at most, Δx is of order $\sqrt{\varepsilon}$ and Δt of order $\sqrt[4]{\varepsilon}$. Note that it cannot prevent Δt and Δx from going to 0 while ε remains fixed.

With the same notations for quantities at times $n\Delta t$ and $(n+1)\Delta t$ as in Section 2, we now focus on the following scheme, corresponding to the

discretization of (1) and (7).

$$(17) \quad u'_i = u_i - \lambda \left(\frac{u_i^2}{2} - \frac{u_{i-1}^2}{2} \right) + \frac{\varepsilon \sigma}{\varrho_i} (u_{i-1} + u_{i+1} - 2u_i),$$

$$(18) \quad \varrho'_i = (1 - \lambda u'_i) \varrho_i + \lambda u'_{i-1} \varrho_{i-1}.$$

Note that (17) is obtained from (7), which is written under a conservative form, as suggested in [13].

If we choose $u^{\text{in}} \equiv 1$, we can note that both upwind and diffusive schemes give $u_i^n = 1$ for any i and n , which is reassuring: in that case, and when ϱ remains nonnegative, the velocity satisfies the Burgers equation, which implies, at least formally, that u remains constant.

Remark 2. The velocity terms which appear in (18) are the ones at time $(n+1)\Delta t$. They must not be at time $n\Delta t$ to ensure the lower bound on ϱ , as we shall see in the proof of Theorem 4 below.

Numerical strategy. Let us here sum up the strategy used to build a relevant numerical solution to the pressureless gases system.

1. Consider 1-periodic initial data.
2. Fix $\varepsilon > 0$ small enough.
3. Regularize $\varrho^{\text{in}}, u^{\text{in}}$ so that they become $C^1(\mathbb{R}; \mathbb{R}_+^*)$ and satisfy (13).
4. Fix Δx and Δt satisfying (15)–(16).
5. Use the numerical scheme (17)–(18).

The previous strategy holds for two reasons. First, the following theorem states that the scheme (17)–(18) is L^∞ -stable, consistent, monotonic, and that it satisfies the OSL condition. Consequently, (ϱ_i^n) and (u_i^n) converge towards ϱ and u , solutions to the viscous pressureless gases system when both Δt and Δx go to 0, ε being fixed. Second, thanks to Theorem 2, the scheme eventually provides a good approximation of a solution to the inviscid pressureless gases system, if one chooses ε small enough, and regularized initial data in $C^1(\mathbb{R}; \mathbb{R}_+^*)$ close to the original ones and satisfying (13). The error between the diffusive numerical and the duality solutions is currently under study, see [10].

Theorem 4. *We assume that (15)–(16) hold. Then we have, for any i and $n \geq 0$,*

$$(19) \quad V \leq u_i^n \leq U,$$

$$(20) \quad u_i^n - u_{i-1}^n \leq \frac{A\Delta x}{1 + An\Delta t},$$

$$(21) \quad \varrho_i^n \geq \frac{R}{1 + An\Delta t} \geq \frac{R}{1 + AT} > 0.$$

Moreover, the discrete total mass is conserved, i.e., for any $n \geq 0$,

$$(22) \quad \sum_i \varrho_i^n \Delta x = \sum_i \varrho_i^0 \Delta x.$$

Finally, when $\varepsilon > 0$ is fixed, the scheme (17)–(18) is consistent with (1) and (7), is first order accurate in time and space, and is monotonic.

Equations (19) and (21) respectively correspond to the maximum principles on the velocity and the density, (20) stands for the discrete version of the OSL condition.

Remark 3. The assumptions (16) on Δt ensure the stability of the scheme. More precisely, the second one is induced by the CFL condition and the third one is similar to standard stability conditions for explicit diffusive schemes. The first one is needed for the required properties of the scheme, as it will be detailed in the proof of Theorem 4.

Proof. We proceed by induction on $n \in \mathbb{N}$, and first investigate the case when $n = 0$. Equations (19) and (21) are obviously satisfied by definitions of U , V and R , and thanks to (15). The fact that (20) holds comes from the fact that u^{in} is smooth, and consequently satisfies the intermediate values inequality.

In the remaining of the proof, we suppose that $A > 0$. The case when $A = 0$ can easily be treated. Let us assume that (19)–(21) hold for a fixed n , and prove them for $n + 1$. We can rewrite Equation (17) as

$$(23) \quad u'_i = \left(1 - \lambda \frac{u_i + u_{i-1}}{2} - \frac{2\varepsilon\sigma}{\varrho_i}\right) u_i + \frac{\varepsilon\sigma}{\varrho_i} u_{i+1} + \left(\lambda \frac{u_i + u_{i-1}}{2} + \frac{\varepsilon\sigma}{\varrho_i}\right) u_{i-1}.$$

Under this form, u'_i is a convex combination of u_{i-1} , u_i and u_{i+1} , since the corresponding coefficients in (23) live in $[0, 1]$ and their sum equals 1. Indeed, we clearly have, thanks to (16) and (21),

$$0 \leq \frac{2\varepsilon\sigma}{\varrho_i} \leq \frac{1}{2},$$

and, thanks to (16) and (19),

$$0 \leq \lambda \frac{u_i + u_{i-1}}{2} \leq \frac{1}{4}.$$

Then it is easy to check that u'_i satisfies (19).

Let us now define, for any i ,

$$\delta_i = u_{i+1} - u_i - \frac{A\Delta x}{1 + An\Delta t},$$

which we know is negative, and prove that δ'_i is also negative, for any i . Thanks to (23), we can write

$$\begin{aligned} u'_{i+1} - u'_i &= \left[1 - \frac{\lambda}{2}(u_{i+1} + u_i) - \frac{\varepsilon\sigma}{\varrho_i} - \frac{\varepsilon\sigma}{\varrho_{i+1}}\right] (u_{i+1} - u_i) \\ &\quad + \frac{\varepsilon\sigma}{\varrho_{i+1}}(u_{i+2} - u_{i+1}) + \left[\frac{\lambda}{2}(u_i + u_{i-1}) + \frac{\varepsilon\sigma}{\varrho_i}\right] (u_i - u_{i-1}). \end{aligned}$$

Then we have

$$\begin{aligned} \delta'_i &= \frac{\varepsilon\sigma}{\varrho_{i+1}}\delta_{i+1} + \left[\frac{\lambda}{2}(u_i + u_{i-1}) + \frac{\varepsilon\sigma}{\varrho_i} - \frac{A\lambda\Delta x}{2(1 + An\Delta t)} \right] \delta_{i-1} \\ &\quad + \left[1 - \frac{\lambda}{2}(u_{i+1} + u_i) - \frac{\varepsilon\sigma}{\varrho_i} - \frac{\varepsilon\sigma}{\varrho_{i+1}} - \frac{A\lambda\Delta x}{2(1 + An\Delta t)} \right] \delta_i \\ &\quad + \frac{A\Delta x}{1 + An\Delta t} \left(1 - \frac{A\Delta t}{1 + An\Delta t} \right) - \frac{A\Delta x}{1 + A(n+1)\Delta t}. \end{aligned}$$

The coefficient before δ_{i+1} is clearly positive. Let us check that the ones before δ_{i-1} and δ_i are positive too. We have

$$\frac{A\lambda\Delta x}{2(1 + An\Delta t)} \leq \frac{\lambda}{2}(u_i + u_{i-1})$$

and

$$\frac{\lambda}{2}(u_{i+1} + u_i) + \frac{\varepsilon\sigma}{\varrho_i} + \frac{\varepsilon\sigma}{\varrho_{i+1}} + \frac{A\Delta t}{2(1 + An\Delta t)} \leq 1,$$

because of (15)–(21). Since the (δ_i) are all negative, we still have to prove that the remaining term is negative to get $\delta'_i \leq 0$. After simplifying by $A\Delta x$, which has no influence on the sign, we write

$$\frac{1 + A(n-1)\Delta t}{(1 + An\Delta t)^2} - \frac{1}{1 + A(n+1)\Delta t} = \frac{-(A\Delta t)^2}{(1 + An\Delta t)^2(1 + A(n+1)\Delta t)},$$

which is clearly negative, and ensures that (20) holds for $n+1$.

We now focus on the properties of ϱ . We successively have, thanks to (21) for n and (20) for $n+1$,

$$\varrho'_i \geq \left[1 - \frac{A\Delta t}{1 + A(n+1)\Delta t} \right] \frac{R}{1 + An\Delta t} = \frac{R}{1 + A(n+1)\Delta t},$$

which concludes the induction. Note that, as we pointed out in Remark 2, if (18) only involved velocities at time $n\Delta t$, the previous inequality would not hold, and we would not get any maximum principle on $1/\varrho$.

We easily notice that in the equality

$$\sum_i \varrho'_i \Delta x = \sum_i \varrho_i \Delta x - \lambda \Delta x \sum_i \varrho_i u'_i + \lambda \Delta x \sum_i \varrho_{i-1} u'_{i-1},$$

the last two terms cancel, which ensures the discrete total mass conservation.

Finally, let us investigate some basic properties of the scheme (17)–(18). The consistency is quite clear. Moreover, if we study u'_i as a function of u_{i-1} , u_i and u_{i+1} , we immediately have

$$\frac{\partial u'_i}{\partial u_{i-1}} = \lambda u_{i-1} + \frac{\varepsilon\sigma}{\varrho_i} \geq 0, \quad \frac{\partial u'_i}{\partial u_i} = 1 - \frac{2\varepsilon\sigma}{\varrho_i} - \lambda u_i \geq 0, \quad \frac{\partial u'_i}{\partial u_{i+1}} = \frac{\varepsilon\sigma}{\varrho_i} \geq 0,$$

which ensures the required property of monotonicity for (17), whereas it is clear for (18).

That ends the proof of Theorem 4. \square

Remark 4. Let us check the behavior of the numerical total momentum. Indeed, in its continuous version (7), the total momentum is conserved, since all the terms besides the time derivative of ϱu are partial derivatives in x . Unfortunately, the scheme does not ensure the exact conservation of the total momentum. Nevertheless, we can write

$$\sum_i q'_i = \sum_i \varrho_i u'_i + \lambda \sum_i \varrho_i u'_i (u'_{i+1} - u'_i),$$

which implies the following inequalities

$$[1 - \lambda(U - V)] \sum_i \varrho_i u'_i \leq \sum_i q'_i \leq \left[1 + \min \left(\frac{1}{n+1}, \lambda(U - V) \right) \right] \sum_i \varrho_i u'_i.$$

Then we have to study the behavior of the quantity

$$\sum_i \varrho_i u'_i = \sum_i q_i - \frac{\lambda}{2} \sum_i \varrho_i (u_i^2 - u_{i-1}^2),$$

for which we have

$$\sum_i q_i - U \min \left(\frac{1}{n}, \lambda(U - V) \right) \sum_i \varrho_i^0 \leq \sum_i \varrho_i u'_i \leq \sum_i q_i + \lambda V(U - V) \sum_i \varrho_i^0.$$

We eventually can write

$$\begin{aligned} \sum_i q'_i &\geq [1 - \lambda(U - V)] \left[\sum_i q_i - U \min \left(\frac{1}{n}, \lambda(U - V) \right) \sum_i \varrho_i^0 \right], \\ \sum_i q'_i &\leq \left[1 + \min \left(\frac{1}{n+1}, \lambda(U - V) \right) \right] \left[\sum_i q_i + \lambda V(U - V) \sum_i \varrho_i^0 \right], \end{aligned}$$

which is not really satisfactory. Nevertheless, since the time and space steps satisfy (16), we have

$$\lambda \leq \frac{R}{4\varepsilon(1 + AT)} \Delta x,$$

which ensures that λ is small when both Δx and Δt go to 0, and $\varepsilon > 0$ is fixed. Of course, that will not prevent the numerical total momentum from varying, but, at least, from one time step to the next one, the variations have to remain small. It is interesting to note that, in the examples of the next section, the total momentum conservation almost holds, meaning that the previous estimates may be improved in some cases.

4. NUMERICAL EXAMPLES

As we already pointed out, a significant drawback of our scheme (17)–(18) is that it does not ensure the exact conservation of the total momentum, since it involves a scheme on the velocity and not on the momentum. Moreover, initial data with vacuum need to be regularized since our scheme cannot stand nil values of ϱ . In this section, apart from checking that the

OSL condition is satisfied (or not, if studying the behavior of the upwind scheme), we shall also study the numerical total momentum.

Of course, we choose the time and space steps in the following tests such that the CFL condition is satisfied when using the upwind scheme, and (15)–(16) when using the diffusive scheme.

4.1. Nil velocity almost everywhere. This test is the one described in Proposition 3 to prove that the OSL condition was eventually not satisfied by the upwind scheme. We choose $\varepsilon = 10^{-6}$. The (regularized) initial data are given by $\varrho^{\text{in}} \equiv 1$ and

$$u^{\text{in}}(x) = \begin{cases} \frac{U + \varepsilon}{2} + \frac{U - \varepsilon}{2} \cos\left(\frac{\pi x}{\sqrt{\varepsilon}}\right) & \text{if } 0 \leq x \leq \sqrt{\varepsilon}, \\ \varepsilon & \text{if } \sqrt{\varepsilon} \leq x \leq 1 - \sqrt{\varepsilon}, \\ \frac{U + \varepsilon}{2} - \frac{U - \varepsilon}{2} \cos\left[\frac{\pi}{\sqrt{\varepsilon}}(x - 1 + \sqrt{\varepsilon})\right] & \text{if } 1 - \sqrt{\varepsilon} \leq x \leq 1, \end{cases}$$

We immediately check that $\min u^{\text{in}} = \varepsilon$, $\max u^{\text{in}} = U$, $\max(u^{\text{in}})' \leq \frac{U\pi}{2\sqrt{\varepsilon}}$ and $\min \varrho^{\text{in}} = 1$. We numerically choose $U = 1$. The space step is set to $\Delta x = 10^{-4}$ on $[0, 1]$, *i.e.* $I = 10^4$, and the Courant number to $\lambda = 0.25$, so that $\Delta t = 2.5 \cdot 10^{-5}$. We perform 100 iterations in time, *i.e.* $T = 2.5 \cdot 10^{-3}$ s. Eventually, it is clear, on Figure 1, that the diffusive scheme is more efficient

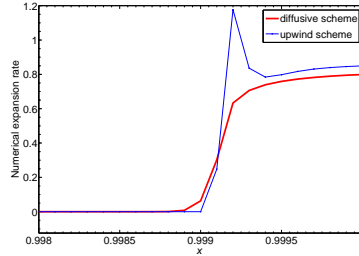


FIGURE 1. Positive part of the numerical expansion rate near 1 at final time T

than the upwind one regarding the OSL condition.

4.2. Piecewise linear velocity. There are other situations when the upwind scheme does not satisfy the OSL condition. For instance, let us consider the following set of initial data

$$(24) \quad \varrho^{\text{in}}(x) = 1, \quad u^{\text{in}}(x) = 1 - x \geq 0, \quad \forall x \in [0, 1),$$

extended by 1-periodicity on \mathbb{R} . In both tests, we choose $T = 1.2$ and $\Delta x = 10^{-4}$.

4.2.1. *Using the upwind scheme.* Using the upwind scheme implies choosing the Courant number λ so that the CFL condition holds. We set $\lambda = 0.1$, which ensures $\lambda \max u < 1$. Then, on Figure 2, the positive part of the numerical expansion rate w is plotted on $[0, 1]$.

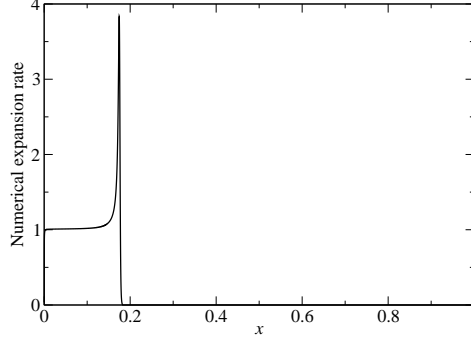


FIGURE 2. “Upwind” plot of w^+ at $t = 0.2$ s with initial data (24)

It is then clear that there are some values of i such that $w_i > 1$, and, in anticipation of the next paragraph, we must point out that, of course, choosing a lower Courant number does not have any effect on the behavior of the numerical expansion rate.

4.2.2. *Using the diffusive scheme.* We choose $\varepsilon = 0.001$. As explained in Section 3, the initial data must be regularized: both ϱ^{in} and u^{in} must be $C^1(\mathbb{R}; \mathbb{R}_+^*)$, and u^{in} is regularized near 0 in order to have a reasonable periodic agreement with the value in 1, and satisfy (13). Since (15)–(16) must hold, it is possible to check that $(\lambda = 0.01, \Delta t = 10^{-6})$ is a relevant choice.

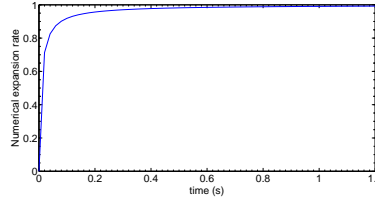


FIGURE 3. “Diffusive” plot of w^+ at $x = 0.1$ with regularized initial data (24)

This time, the OSL condition is satisfied, as one can see on Figure 3 at $x = 0.1$, where the upwind scheme experiences trouble with the expansion rate for times smaller than 0.2.

Eventually, to investigate the total numerical momentum, on Figure 4, we show its behavior with respect to t , till T , and the result is quite convincing. On the same figure, we also show the total numerical mass, which is of course exactly conserved.

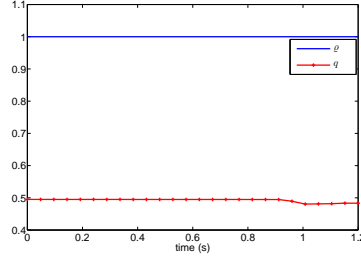


FIGURE 4. Numerical total mass and momentum

4.3. Continuous velocity and piecewise constant density. Of course, the upwind scheme may often provide a numerical solution satisfying the OSL condition. It is then interesting to check the behavior of both upwind and diffusive schemes, which should be similar. We consider the following initial data for the density

$$\varrho^{\text{in}}(x) = 1, \quad 0 \leq x < 0.2, \quad \varrho^{\text{in}}(x) = 0.5, \quad 0.2 \leq x < 1,$$

and for the velocity

$$u^{\text{in}}(x) = 0.5(1 - \cos(10\pi x)), \quad 0 \leq x < 1,$$

extended by 1-periodicity on \mathbb{R} . The final time is $T = 2$.

For the diffusive scheme, we pick $\varepsilon = 10^{-12}$. Since $\min \varrho^{\text{in}} = 0.5$, the regularization of the initial density can be chosen not depending on ε . Since u^{in} is already C^1 , we need no regularization, but we have to add a nonnegative term to ensure that $\min u^{\text{in}} > 0$, for instance, $V = 0.032$. And we note that $\max(u^{\text{in}})' = 5\pi$.

Then we choose $\Delta x = 0.002$, and $\lambda = 0.1$ for both upwind and diffusive cases. The space step satisfies (15), as required, since $\Delta x \leq \frac{2V}{5\pi}$.

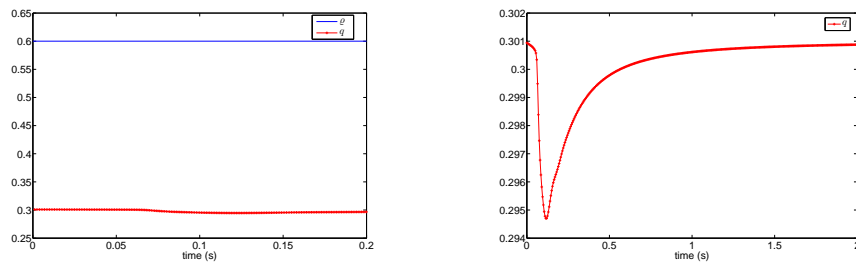


FIGURE 5. (a) Numerical total mass and momentum computed with the diffusive scheme, (b) zoom on the total momentum

First, we check on Figure 5a–b that the numerical total momentum is still well conserved by the diffusive scheme.

Let us get into some more details of the behavior of both schemes with respect to time. For small times, one can check on Figures 6–7 that both schemes give very similar results for ρ , u and w . If we accurately study Figure 6b, we can see that the upwind scheme has very small variations with respect to the diffusive scheme near some points, which are in fact the jump points of the density, see Figure 8a.

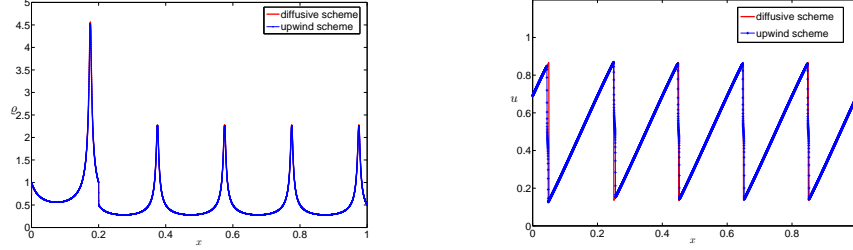


FIGURE 6. (a) Density at 0.04s, (b) velocity at 0.2s

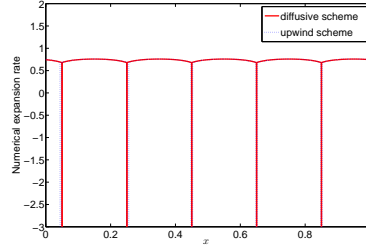


FIGURE 7. Numerical expansion rate at time 0.2s

Hence, when time grows, the behaviors of both schemes become more and more different, as seen on Figures 8–10, for quite small times for the density, later for the velocity and the numerical expansion rate.

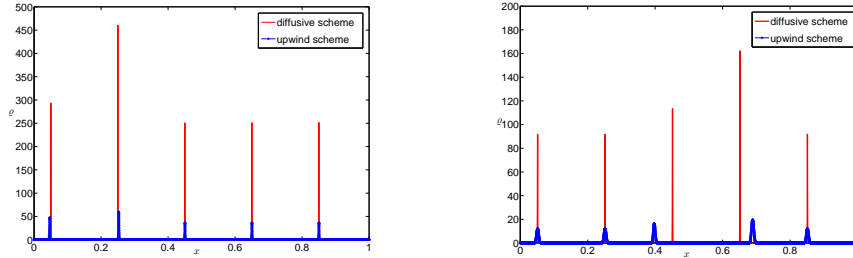


FIGURE 8. Density at times (a) 0.2s, and (b) 1s

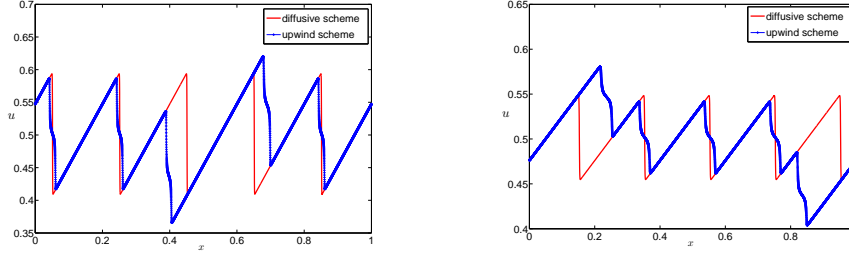


FIGURE 9. Velocity at times (a) 1 s, and (b) 2 s

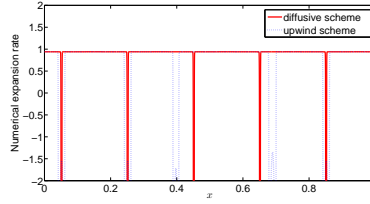


FIGURE 10. Numerical expansion rate at time 1 s

It is important to note that the numerical expansion rates are still upper-bounded by 1, for both schemes. The differences between the numerical solutions is consequently not related to the OSL condition. In fact, we believe that the diffusive scheme is more trustworthy. Indeed, the upwind scheme has natural numerical diffusion, which is responsible for the variations. This numerical diffusion seems to be fully avoided by the diffusive scheme: it is absorbed by the artificial viscosity inserted in the scheme, and its effect cannot numerically appear.

Acknowledgements. L. Boudin wants to thank F. James and R. Eymard for the various discussions they shared about the topic of this article a few years ago.

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